

Calculus III: Coursework 3
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The deadline is 5PM on Tuesday Oct 17th. Hand-in is to the green box in the basement of the Maths building.

1. The problem is to find the curve of length ℓ which is such that the area enclosed between the curve and the x axis is a maximum, where one end of the curve is fixed at the origin and the other end is on the x axis. Solve this as follows.

(a) Using the calculus of variations show that the curve is part of a circle. [You may find a substitution of the form $y - b = \lambda \cos \theta$ useful in solving the differential equation that arises.]

(b) Writing this circle as

$$(x - a)^2 + (y - b)^2 = \lambda^2, \quad (1)$$

find the relations implied by the fact that the origin is on the circle (at a value $\theta = \theta_0$ say) and that the length round the curve back to the x axis is ℓ , expressing the latter in terms of θ_0 . Note that you may assume that $b \geq 0$, i.e. that the centre is in $y \geq 0$, by symmetry about $y = 0$.

(c) Sketch the circle and use its geometry to write down the area. Then maximize this. [You will find the derivative is always negative, and can use the lower bound on θ_0 obtained by taking $b \geq 0$.]

[Note: it is possible to do parts (b) and (c) starting by assuming equation (1). Beware that the version in Dr. Saha's old notes is not correct.] [40]

Answer

(a) The action is $\int (y + \lambda \sqrt{1 + y'^2}) dx$.

This is independent of x and the resulting Hamiltonian is

$$-y - \frac{\lambda}{\sqrt{1 + y'^2}} = -b$$

where b is a constant (the minus is for convenience: the left side has to be negative). Thus

$$\begin{aligned} -\frac{\lambda}{\sqrt{1 + y'^2}} &= (y - b) \\ \Rightarrow \frac{\lambda^2}{(y - b)^2} &= 1 + y'^2 \\ \Rightarrow \int \frac{y - b}{\sqrt{\lambda^2 - (y - b)^2}} dy &= \int dx \end{aligned}$$

Let $y - b = \lambda \cos \theta$ and get

$$\begin{aligned} \int dx &= \int \frac{-\lambda \sin \theta \lambda \cos \theta}{\lambda \sin \theta} d\theta = \int -\lambda \cos \theta d\theta \\ \Rightarrow x - a &= -\lambda \sin \theta \text{ for some constant } a \end{aligned}$$

and hence

$$(x - a)^2 + (y - b)^2 = \lambda^2,$$

which is a circle with centre at (a, b) and radius λ .

(b) For $(0, 0)$ to be on the circle we need $a^2 + b^2 = \lambda^2$.

If the origin is at θ_0 (where $\lambda \sin \theta_0 = a$) the other point where the circle meets the x -axis is at $(2a, 0)$, at the point where $\theta = -\theta_0$. The length round the circle is thus $2\lambda\theta_0 = \ell$.

(c) The area is the sum of the area of the segment of the circle with total angle $2\theta_0$, which is $\pi\lambda^2(2\theta_0/2\pi) = \lambda^2\theta_0$ and the area of the triangle with base $2a$ and height b , which gives in total an area

$$\begin{aligned} A &= \lambda^2\theta_0 + ab = \lambda^2\theta_0 - \lambda^2 \sin \theta_0 \cos \theta_0 \\ &= \lambda^2(\theta_0 - \tfrac{1}{2} \sin(2\theta_0)) \\ &= \frac{\ell^2}{4\theta_0^2} \left(\theta_0 - \tfrac{1}{2} \sin(2\theta_0) \right) \end{aligned}$$

This gives

$$\begin{aligned} \frac{dA}{d\theta_0} &= \tfrac{1}{4}\ell^2 \left(\frac{-(1 + \cos(2\theta_0))}{\theta_0^2} + \frac{\sin 2\theta_0}{\theta_0^3} \right) \\ &= \frac{2\ell^2 \cos^2 \theta_0}{4\theta_0^3} (\tan \theta_0 - \theta_0) \end{aligned}$$

which, since $\pi \geq \theta_0 \geq \pi/2$, is always ≤ 0 . Hence the maximum is when θ_0 reaches its lowest allowed value i.e. $\pi/2$. The curve is then a semicircle.

Marks (a) Lagrangian with λ 5, Hamiltonian 4, the rearranged DE 4, solving it 4 (only a few marks for solving because of the hint and the stated answer), (b) method 4 accuracy 4, (c) correct area 6, in form of function of θ_0 only 3, derivative 4, final answer 2 (again, few marks for this bit because of the hint).

2. Work out the angle between the vectors $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j} - \mathbf{k}$. [10]

Answer

The dot product is 9 while the magnitudes are 3 and $3\sqrt{2}$, which gives $\pi/4$ or 45° .

Marks: Knowing dot-product is involved and using it (4), correct calculation (6).

3. Describe each of the following surfaces in three dimensional space. [For example, $x^2 + y^2 = 1$ is a cylinder of radius 1 whose axis is the z axis.]

(a) $z = 0$

(b) $x^2 + y^2 + z^2 - 2y - 4z + 1 = 0$

(c) $x^2 + y^2 = z$

Answer

Surfaces are as follows.

- (a) The x, y plane.
- (b) A sphere of radius 2, centred at $(0, 1, 2)$.
- (c) A bowl-shaped surface above the x, y plane (paraboloid).

Marks: Plane (4), sphere (6), paraboloid (8).

4. Consider a sphere of radius 3 centred at the origin.

- (a) Express the surface of the sphere in the form $f(x, y, z) = 0$.
- (b) Find a plane tangent to the sphere at the point $\mathbf{r}_1 = (1, 2, 2)$.
- (c) Find the position vector of points on the line through $\mathbf{r}_1$ and $\mathbf{r}_2 = (2, 1, 2)$.
- (d) Find the position vector of points on a line through $\mathbf{r}_1$ and normal to the sphere.

Answer

We have

(a) $x^2 + y^2 + z^2 - 9 = 0$

(b) Taking the gradient at $\mathbf{r}_1$ gives a vector normal to the surface

$$\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

The tangent plane is $(\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{p} = 0$. Thus

$$[(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 2)\mathbf{k}] \cdot [\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}] = 0$$

or

$$x + 2y + 2z = 9.$$

Alternatively, use the fact that writing a surface as $z = f(x, y)$ the tangent plane at (x_0, y_0, z_0) is

$$z - z_0 = (x - x_0) \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} + (y - y_0) \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)},$$

which was done in Calculus II.

(c) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} - \mathbf{j})$ where $-\infty < s < \infty$.

(d) $\mathbf{r} = s(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ where $-\infty < s < \infty$.

(Normal direction can be inferred from the answer to (b) if (b) was not calculated that way.)

Marks: Parts a–d (4 each), plus correct gradient or partial derivatives in part (b) another (4).

5. For any vector $\mathbf{v}$ (having nonzero magnitude v) relate dv/dt to $d\mathbf{v}/dt$ and hence show that

$$dv/dt = 0 \Rightarrow d\mathbf{v}/dt \perp \mathbf{v} .$$

[12]

Answer

Since $v^2 = \mathbf{v} \cdot \mathbf{v}$ we have

$$v \frac{dv}{dt} = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} .$$

If the LHS is zero we must have

$$\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 0$$

which is the result.

Marks: Knowing what v means (4), knowing that $\perp$ means zero dot product (4), differentiating correctly (4).